

Natural Numbers via Peano Axioms

Foundational Construction of Arithmetic from First Principles

Table of contents

Introduction	2
1 Motivation	3
1.1 The Counting Problem	3
1.2 Historical Failures	3
1.2.1 Concrete Counting (Egyptian, ~3000 BCE)	3
1.2.2 Geometric Representation (Greek, ~500 BCE)	4
1.2.3 Intuitive Infinity (Pre-Cantor, ~1850)	4
1.3 Peano's Breakthrough (1889)	4
2 Intuitive Understanding	5
2.1 Three Mental Models	5
2.1.1 Model 1: Chain of Beads	5
2.1.2 Model 2: Stepladder	5
2.1.3 Model 3: Construction Process	6
2.2 The Core Principle	6
2.3 Edge Cases and Counterexamples	7
2.3.1 What if we had cycles?	7
2.3.2 What if elements were unreachable from 0?	7
3 The Peano Axioms	8
3.1 Axiom 1: Existence	8
3.2 Axiom 2: Closure	8
3.3 Axiom 3: Injectivity	9
3.4 Axiom 4: Zero is Not a Successor	9
3.5 Axiom 5: Induction Principle	10

4 Understanding the Axioms	11
4.1 Why Each Axiom is Necessary	11
4.1.1 Without Axiom 1 (Existence)	11
4.1.2 Without Axiom 2 (Closure)	11
4.1.3 Without Axiom 3 (Injectivity)	11
4.1.4 Without Axiom 4 (Zero Not Successor)	12
4.1.5 Without Axiom 5 (Induction)	12
4.2 Connecting Formal to Intuitive	13
5 Key Properties	14
5.1 Property 1: Zero is Unique	14
5.2 Property 2: Every Non-Zero Element is a Successor	15
5.3 Property 3: Successor is Bijection onto $\mathbb{N} \setminus \{0\}$	15
5.4 Property 4: \mathbb{N} is Infinite	16
5.5 Property 5: Successor Has No Fixed Points	17
6 Axioms vs Definitions vs Theorems	18
6.1 Three Types of Mathematical Statements	18
6.1.1 Axioms (The Foundation)	18
6.1.2 Definitions (Our Choices)	18
6.1.3 Theorems (What We Prove)	19
6.2 The Framework	19
7 Defining Addition	20
7.1 The Problem	20
7.2 Recursive Definition Strategy	20
7.3 Definition of Addition	20
7.3.1 Base Case	21
7.3.2 Recursive Case	21
7.4 How the Definition Works	21
7.4.1 Example 1: Computing $3 + 2$	21
7.4.2 Example 2: Computing $0 + 3$	22
7.5 A Shocking Truth	23
7.6 What's Next	23

Introduction

This document presents a rigorous construction of the natural numbers from axiomatic foundations. We develop the complete theory of \mathbb{N} starting only from the Peano Axioms, proving even seemingly “obvious” properties through careful logical reasoning.

By the end of this treatment, you will be able to:

- State and explain all five Peano Axioms precisely
- Understand why each axiom is necessary through counterexamples
- Distinguish between axioms, definitions, and theorems
- Prove basic properties using mathematical induction
- Define arithmetic operations recursively
- Recognize and debug common errors in inductive proofs

Prerequisites: Basic familiarity with sets, functions, and logical reasoning. We assume knowledge of injectivity (different inputs produce different outputs) and the distinction between necessary and sufficient conditions.

Structure: We begin with historical motivation, build intuitive understanding through multiple mental models, state the formal axioms, examine why each is necessary, derive key properties, and finally construct addition from first principles.

1 Motivation

1.1 The Counting Problem

Consider ancient mathematicians attempting to answer basic questions:

- “How many sheep do I have?”
- “Is my flock larger than yours?”
- “If I combine our flocks, how many total?”

They had available:

- Physical objects (pebbles, fingers, tally marks)
- The ability to match objects one-to-one
- The concept of “one more”

What was difficult: **How do you define “number” without using numbers?** How do you ensure everyone means the same thing by “three”? How do you prove $3 + 2 = 2 + 3$ without just “seeing” it?

1.2 Historical Failures

1.2.1 Concrete Counting (Egyptian, ~3000 BCE)

The Egyptians used different symbols for 1, 10, 100, 1000.

Problem: No systematic way to extend indefinitely. What comes after the largest symbol invented?

1.2.2 Geometric Representation (Greek, ~500 BCE)

Numbers were represented as lengths and areas.

Problem: Cannot represent irrational numbers as ratios. The discovery that $\sqrt{2}$ exists geometrically but not arithmetically in their system led to crisis.

1.2.3 Intuitive Infinity (Pre-Cantor, ~1850)

Mathematicians worked with vague notions that “numbers go on forever.”

Problem: Led to paradoxes about infinite sets. Questions like “Is $\infty + 1 = \infty$?” had no clear answer.

1.3 Peano's Breakthrough (1889)

Giuseppe Peano asked: **What is the absolute minimum we must assume to make arithmetic work?**

His key insight: We don't need to define what numbers “are”—we only need to specify **how they behave**.

Rejected approach: - Trying to define “three” as “the symbol after two” (circular!) - Trying to define numbers as sets of objects (conflates number with quantity)

Successful approach:

1. Start with one primitive object: **0** (or 1 in original formulation)
2. Define one primitive operation: **successor** $S(n) =$ “the next number”
3. State axioms characterizing how successor behaves
4. Everything else (addition, multiplication, exponentiation) derives from these axioms

This approach revolutionized mathematics by showing how to axiomatize mathematical structures. It inspired:

- Zermelo-Fraenkel axioms for set theory
- Hilbert's axioms for geometry
- Modern abstract algebra

2 Intuitive Understanding

Before seeing the formal axioms, we develop three equivalent mental models for the natural numbers.

2.1 Three Mental Models

2.1.1 Model 1: Chain of Beads

0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow ...
S S S S S

Imagine an infinite chain of beads:

- First bead is labeled **0** (starting point)
- Each bead has exactly one string attached leading to the next bead (successor)
- No bead has a string leading back to 0 (no cycles)
- The chain continues forever (infinite)
- You can reach any bead by starting at 0 and following strings

Key features:

- **Linear** (not branching, not circular)
- **Infinite** (no last bead)
- **Connected** (every bead reachable from 0)
- **Directed** (arrows go one way)

2.1.2 Model 2: Stepladder

...
3
2
1
Ground 0

Imagine an infinite ladder reaching upward:

- Ground level = 0 (you start here)

- Each rung is one step up (successor function)
- You can always climb one more rung (no highest step)
- You can't skip rungs (must go $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$, not $0 \rightarrow 5$ directly)
- You can never climb below ground level (no negative rungs in \mathbb{N})

This model emphasizes:

- **Well-ordered:** Clear notion of “before” and “after”
- **Discrete steps:** No “between” 3 and 4 (unlike real numbers)
- **Grounded:** Everything builds from 0

2.1.3 Model 3: Construction Process

```

Stage 0: {0}
Stage 1: {0, 1}           where 1 = S(0)
Stage 2: {0, 1, 2}         where 2 = S(1) = S(S(0))
Stage 3: {0, 1, 2, 3}      where 3 = S(2) = S(S(S(0)))
...
Stage : = {0, 1, 2, 3, ...}

```

Think of natural numbers as **iterative construction**:

- We start with just 0
- At each stage, we add the successor of the largest element so far
- Continue this process infinitely
- \mathbb{N} is exactly what this process generates

This model emphasizes:

- **Generative:** Numbers are built from simpler numbers
- **Minimal:** \mathbb{N} contains only what this process generates (nothing extra)
- **Recursive:** Each stage uses the previous stage

2.2 The Core Principle

All three models reveal the same fundamental insight: **recursive structure**.

To define what happens at stage n , we refer to stage $n - 1$. This recursive pattern manifests as:

- **Inductive definition** (define complex things from simpler things)
- **Structural recursion** (build on previous structure)
- **Mathematical induction** (prove for n by assuming true for $n - 1$)

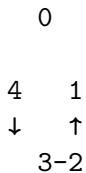
This recursive structure is why the Peano axioms work and why mathematical induction is the natural proof technique for \mathbb{N} .

2.3 Edge Cases and Counterexamples

These mental models help us predict what properties the formal axioms must satisfy.

2.3.1 What if we had cycles?

Consider clock arithmetic modulo 5:



Where $S(4) = 0$ (loops back). This violates the chain/ladder/construction models:

- Chain: String leads back to first bead
- Ladder: Can climb back down to ground
- Construction: Process doesn't generate anything new after 5 steps

This structure satisfies some natural properties but lacks others. Distinguishing \mathbb{N} from finite cycles requires an explicit axiom.

2.3.2 What if elements were unreachable from 0?

Consider $\mathbb{N} \cup \{\infty\}$ where ∞ is isolated:

$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$ (reachable chain)

∞ (unreachable element)

This violates the construction model: ∞ is not generated by repeatedly applying successor to 0.

An axiom ensuring **minimality**—that \mathbb{N} contains exactly what's reachable from 0—is necessary.

3 The Peano Axioms

We now make our intuitive understanding precise. The natural numbers are characterized by:

- A set \mathbb{N}
- A distinguished element $0 \in \mathbb{N}$
- A function $S : \mathbb{N} \rightarrow \mathbb{N}$ (the successor function)

These must satisfy five axioms.

3.1 Axiom 1: Existence

$$0 \in \mathbb{N}$$

In words: Zero is a natural number.

This ensures \mathbb{N} is non-empty and provides a starting point for building all other numbers.

Type information:

- 0 is an element (not a set, not a function)
- \mathbb{N} is a set
- \in is the membership relation

3.2 Axiom 2: Closure

$$\forall n \in \mathbb{N}, \quad S(n) \in \mathbb{N}$$

In words: The successor of any natural number is also a natural number.

This ensures we can keep building—there’s no edge where S fails. The set \mathbb{N} is “closed” under the successor operation.

Type information:

- $n \in \mathbb{N}$ (input is a natural number)
- $S(n) \in \mathbb{N}$ (output is a natural number)
- $S : \mathbb{N} \rightarrow \mathbb{N}$ is a total function

This axiom also implicitly asserts S is **single-valued** (function property): each n has exactly one $S(n)$, not multiple.

3.3 Axiom 3: Injectivity

$$\forall n, m \in \mathbb{N}, \quad S(n) = S(m) \implies n = m$$

Contrapositive form (equivalent):

$$\forall n, m \in \mathbb{N}, \quad n \neq m \implies S(n) \neq S(m)$$

In words: Different natural numbers have different successors.

This prevents “collisions”—situations where two different numbers have the same successor.

Distinction from function property:

- **Function (Axiom 2):** Each input has exactly **one** output
- **Injective (Axiom 3):** Different inputs have **different** outputs

All injective functions are functions, but not all functions are injective.

Visual representation:

Not injective:

3 → 5 (collision – both map to 5)
4

Injective:

3 → 5
4 → 6 (different outputs)

3.4 Axiom 4: Zero is Not a Successor

$$\forall n \in \mathbb{N}, \quad S(n) \neq 0$$

Equivalently: $\neg \exists n \in \mathbb{N}$ such that $S(n) = 0$

In words: Zero is not the successor of any natural number.

This prevents cycles and makes 0 the unique minimal element. Without this axiom, we cannot distinguish infinite chains (like \mathbb{N}) from finite cycles (like clock arithmetic).

Combined with Axiom 3: These two axioms together ensure no cycles anywhere in the structure, not just at 0.

3.5 Axiom 5: Induction Principle

Set-theoretic form:

Let $P \subseteq \mathbb{N}$. If:

1. $0 \in P$, and
2. $\forall n \in \mathbb{N}, n \in P \implies S(n) \in P$

then $P = \mathbb{N}$.

Predicate form:

Let $P(n)$ be a property of natural numbers. If:

1. $P(0)$ is true, and
2. $\forall n \in \mathbb{N}, P(n) \implies P(S(n))$

then $P(n)$ is true for all $n \in \mathbb{N}$.

In words: If a set P contains 0, and whenever P contains n it also contains $S(n)$, then P contains all natural numbers.

Two purposes:

1. **Proof technique:** Gives us mathematical induction as a method
2. **Minimality (more fundamental):** Ensures \mathbb{N} is the **minimal** set satisfying Axioms 1-4

The induction axiom prevents “ghost elements” that are not reachable from 0 by repeatedly applying successor. It ensures:

$$\mathbb{N} = \{0, S(0), S(S(0)), S(S(S(0))), \dots\}$$

and nothing more.

Domino analogy:

- Condition (i): First domino falls
- Condition (ii): If domino n falls, then domino $S(n)$ falls
- Conclusion: All dominoes fall

4 Understanding the Axioms

4.1 Why Each Axiom is Necessary

We examine what breaks when each axiom is removed or violated.

4.1.1 Without Axiom 1 (Existence)

Violating structure: Empty set \emptyset

What breaks: No starting point. Cannot begin building anything.

4.1.2 Without Axiom 2 (Closure)

Violating structure: $S(3)$ is undefined, or $S(3) \notin \mathbb{N}$

What breaks:

- Cannot always compute successor
- Induction fails (cannot proceed from n to $S(n)$ if $S(n)$ doesn't exist in \mathbb{N})
- Arithmetic becomes impossible: What is $3 + 1$ if $S(3)$ is undefined?

Example:

$$\begin{array}{ccccccc} 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3 \\ & & & & & & \downarrow \\ & & & & & & \text{(dead end, } S(3) \text{ undefined)} \end{array}$$

4.1.3 Without Axiom 3 (Injectivity)

Violating structure: Colliding paths

$$\begin{array}{ccccccc} 0 & \rightarrow & 1 & \rightarrow & 2 & & \\ & & & & & & \downarrow \\ 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 \rightarrow \dots \\ & & & & & & \uparrow \end{array}$$

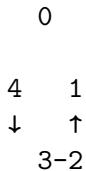
Where $S(2) = 5$ and $S(4) = 5$ (collision at 5)

What breaks:

1. **Addition ill-defined:** If we define $n + 1 = S(n)$, then $2 + 1 = S(2) = 5$, but also $4 + 1 = S(4) = 5$. This implies $2 + 1 = 4 + 1$, so $2 = 4$. Contradiction!
2. **No unique representation:** The number 5 can be written as $S(2)$ or $S(4)$. Which predecessor is “real”?
3. **Cannot invert S uniquely:** If $S(n) = 7$, is $n = 6$ or something else? Ambiguous!

4.1.4 Without Axiom 4 (Zero Not Successor)

Violating structure: Clock arithmetic $\mathbb{Z}/5\mathbb{Z}$



Where $S(4) = 0$ (loops back)

What breaks:

1. **No first element:** Every element has a predecessor. Cannot identify a unique starting point for induction.
2. **Cannot distinguish finite from infinite:** This structure is finite (5 elements), but without Axiom 4, we cannot rule it out as a model of \mathbb{N} .
3. **Arithmetic differs:** In $\mathbb{Z}/5\mathbb{Z}$, we have $3 + 2 = 0$ (modular arithmetic), whereas in \mathbb{N} , we have $3 + 2 = 5$ (standard arithmetic).
4. **Well-ordering fails:** In \mathbb{N} , every non-empty subset has a least element. In circular structures, no minimum exists.

4.1.5 Without Axiom 5 (Induction)

Violating structure: $\mathbb{N} \cup \{\infty\}$ where $S(\infty) = \infty$

$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \dots$ (reachable from 0)
 ∞ (isolated, unreachable)

Check Axioms 1-4:

- Axiom 1: 0 exists
- Axiom 2: $S(\infty) = \infty \in \mathbb{N} \cup \{\infty\}$
- Axiom 3: S is injective (different elements map to different successors)
- Axiom 4: No element maps to 0

Axiom 5 fails:

Let $P = \{0, 1, 2, 3, \dots\}$ (standard naturals, excluding ∞)

- (i) $0 \in P$
- (ii) If $n \in P$, then $S(n) \in P$

But $P \neq \mathbb{N} \cup \{\infty\}$ because $\infty \notin P$!

What breaks:

1. **Non-uniqueness:** Which structure is THE natural numbers? \mathbb{N} alone? $\mathbb{N} \cup \{\infty\}$? $\mathbb{N} \cup \{\omega, \omega + 1, \omega + 2, \dots\}$?
2. **Induction as proof technique fails:** We can prove properties for $\{0, 1, 2, \dots\}$ but not for ghost elements.
3. **Recursive definitions don't work on ghost elements:** If we recursively define operations, they only apply to reachable elements.

4.2 Connecting Formal to Intuitive

Return to our three mental models:

Chain of Beads:

- Axiom 1: First bead exists (0)
- Axiom 2: Every bead has a string to next bead (closure)
- Axiom 3: No two beads connect to same next bead (injective)
- Axiom 4: No string connects back to first bead (no cycle)
- Axiom 5: Chain contains only beads reachable from first (minimality)

Ladder:

- Axiom 1: Ground level exists
- Axiom 2: Every rung has a rung above it
- Axiom 3: No two rungs lead to same upper rung
- Axiom 4: No rung leads down to ground
- Axiom 5: Ladder contains only rungs reachable from ground

Construction:

- Axiom 1: Process starts with $\{0\}$
- Axiom 2: Process can always add successor
- Axiom 3: New element is distinct from all previous
- Axiom 4: Process never returns to 0
- Axiom 5: \mathbb{N} is exactly what process generates

The formal axioms precisely capture these intuitive pictures.

5 Key Properties

We now derive fundamental properties from the axioms alone. Even “obvious” facts require proof.

5.1 Property 1: Zero is Unique

Theorem: There is exactly one element that is not a successor.

Proof:

We know from Axiom 4 that 0 is not a successor.

Uniqueness: Suppose x is another element that’s not a successor, with $x \neq 0$.

Consider the set $P = \{0, S(0), S(S(0)), S(S(S(0))), \dots\}$ —all elements reachable from 0.

Check induction conditions:

- (i) $0 \in P$ (by definition of P)
- (ii) If $n \in P$, then $S(n) \in P$ (by construction of P)

By Axiom 5 (induction): $P = \mathbb{N}$

But $x \notin P$ (since x is not a successor and $x \neq 0$, so x is not reachable from 0).

This contradicts $P = \mathbb{N}$.

Therefore, no such x exists. **Zero is the unique minimal element.**

5.2 Property 2: Every Non-Zero Element is a Successor

Theorem: For all $n \in \mathbb{N}$, if $n \neq 0$, then $n = S(m)$ for some $m \in \mathbb{N}$.

Proof by Induction:

Let $P = \{0\} \cup \{n \in \mathbb{N} : n = S(m) \text{ for some } m\}$

In words: P consists of 0 and all successors.

Claim: $P = \mathbb{N}$

Base case: $0 \in P$ (by definition of P)

Inductive step: Suppose $n \in P$. Then $S(n) \in P$ because $S(n) = S(n)$ (it's a successor)

By Axiom 5 (induction): $P = \mathbb{N}$

Therefore, every element is either 0 or a successor.

5.3 Property 3: Successor is Bijection onto $\mathbb{N} \setminus \{0\}$

Theorem: $S : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ is a bijection.

Proof:

Injective: Axiom 3 directly states S is injective.

Surjective onto $\mathbb{N} \setminus \{0\}$: Property 2 states every non-zero element is a successor.

Therefore S is a bijection from \mathbb{N} to $\mathbb{N} \setminus \{0\}$.

Consequence: We can define a **predecessor function** $P : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$:

$$P(n) = \text{the unique } m \text{ such that } S(m) = n$$

This exists and is unique by the bijection property.

Note: P is only defined on $\mathbb{N} \setminus \{0\}$, not on 0 itself (since 0 has no predecessor).

5.4 Property 4: \mathbb{N} is Infinite

Theorem: \mathbb{N} has no largest element.

Proof by Contradiction:

Assume, for contradiction, that \mathbb{N} has a largest element k .

Then $\mathbb{N} = \{0, 1, 2, \dots, k\}$ for some $k \in \mathbb{N}$.

Apply Axiom 2 (Closure): Since $k \in \mathbb{N}$, we have $S(k) \in \mathbb{N}$ by Axiom 2.

Derive contradiction:

We show $S(k)$ cannot equal any element of $\{0, 1, 2, \dots, k\}$:

- $S(k) \neq 0$ by Axiom 4
- $S(k) \neq 1, 2, \dots, k$ because these are $S(0), S(1), \dots, S(k-1)$ respectively, and S is injective (Axiom 3). If $S(k) = S(i)$ for $i < k$, then $k = i$ by injectivity, contradicting $k > i$.

Therefore $S(k) \notin \{0, 1, 2, \dots, k\} = \mathbb{N}$.

But this contradicts Axiom 2, which states $S(k) \in \mathbb{N}$.

Conclusion: Our assumption was false. \mathbb{N} has no largest element.

Alternative formulation:

By induction, the first $k + 1$ elements generated from 0 are distinct:

$$0, S(0), S^2(0), \dots, S^k(0)$$

are all different (by injectivity and Axiom 4). If \mathbb{N} had exactly $k + 1$ elements, then:

$$\mathbb{N} = \{0, S(0), S^2(0), \dots, S^k(0)\}$$

But then $S(S^k(0)) = S^{k+1}(0)$ must equal one of these elements. It cannot equal 0 (Axiom 4), and it cannot equal $S^i(0)$ for $i < k$ (by injectivity). Contradiction!

5.5 Property 5: Successor Has No Fixed Points

Theorem: For all $n \in \mathbb{N}$, $S(n) \neq n$.

Proof by Induction:

Let $P(n)$ be the statement “ $S(n) \neq n$ ”.

Base case ($n = 0$):

By Axiom 4, $S(n) \neq 0$ for all $n \in \mathbb{N}$.

In particular, $S(0) \neq 0$.

So $P(0)$ is true.

Inductive step:

Inductive Hypothesis (IH): Assume $S(k) \neq k$ for some $k \in \mathbb{N}$.

Goal: Prove $S(S(k)) \neq S(k)$.

Proof by contradiction:

Suppose $S(S(k)) = S(k)$.

By Axiom 3 (injectivity): If $S(a) = S(b)$, then $a = b$.

Applying this with $a = S(k)$ and $b = k$:

$$S(S(k)) = S(k) \implies S(k) = k$$

But this contradicts our inductive hypothesis that $S(k) \neq k$!

Therefore our assumption was false: $S(S(k)) \neq S(k)$.

Conclusion:

By the principle of mathematical induction (Axiom 5):

- $P(0)$ is true
- For all k , $P(k) \implies P(S(k))$

Therefore $P(n)$ is true for all $n \in \mathbb{N}$.

That is, $S(n) \neq n$ for all $n \in \mathbb{N}$.

6 Axioms vs Definitions vs Theorems

Before defining arithmetic operations, we must understand a fundamental distinction in mathematics.

6.1 Three Types of Mathematical Statements

6.1.1 Axioms (The Foundation)

What they are: Statements we **accept without proof** as the starting point.

Example: Axiom 4: $S(n) \neq 0$ for all $n \in \mathbb{N}$

Key properties:

- Cannot be proven from anything more basic
- We simply **agree** these are true
- They define the “rules of the game”

Analogy: Like rules of chess—“bishops move diagonally” isn’t proven; it’s the rule we accept.

6.1.2 Definitions (Our Choices)

What they are: We **introduce new concepts** by specifying what they mean in terms of things we already have.

Example we’ll see soon:

$$n + 0 = n$$

Key properties:

- Not proven from axioms
- Not axioms themselves
- **Choices** we make about what new symbols mean
- We could choose differently (but we choose wisely!)

Analogy: Like defining a new chess piece. “The dragon moves like a knight but can also jump two squares diagonally”—we’re inventing this, not discovering it.

Why definitions matter:

- We have \mathbb{N} , 0, and S from axioms
- But we don’t have “+” yet!

- We must **define** what “+” means using only 0 and S
- The definition is our choice (but a smart choice!)

6.1.3 Theorems (What We Prove)

What they are: Statements that **follow logically** from axioms + definitions.

Example we'll prove: $m + n = n + m$ (commutativity)

Key properties:

- Must be **proven** using axioms, definitions, and logic
- Not obvious (even if it feels obvious!)
- Requires rigorous proof

Analogy: In chess, “with only a king and bishop, you cannot checkmate a lone king”—this is a theorem provable from the rules.

6.2 The Framework

```

AXIOMS (given, unprovable foundation)
  ↓
DEFINITIONS (we choose these using axioms)
  ↓
THEOREMS (we prove these using axioms + definitions)

```

Example from Peano arithmetic:

```

AXIOM:  $S(n) \neq 0$  for all  $n$ 
  ↓
DEFINITION:  $n + 0 = n$ 
             $n + S(m) = S(n + m)$ 
  ↓
THEOREM:  $m + n = n + m$  (commutativity - must be proven!)

```

Critical distinction:

- “ $n + 0 = n$ ” is a **definition** (not from Axiom 4!)
- “ $m + n = n + m$ ” is a **theorem** (not from Axiom 3!)

The axioms give us structure. Definitions let us build new operations. Theorems reveal properties that follow necessarily.

7 Defining Addition

We now construct addition from first principles.

7.1 The Problem

We have:

- \mathbb{N} (set of natural numbers)
- 0 (the starting element)
- S (successor function)

We want:

- “+” (addition operation)

But “+” doesn’t exist yet! We must create it using only what the axioms give us.

7.2 Recursive Definition Strategy

Key insight: We cannot define all additions at once (infinitely many pairs!).

Solution: Define addition **recursively**:

- Give a rule for the simplest case (base case: adding 0)
- Give a rule for reducing complex cases to simpler cases (recursive case: adding $S(m)$ in terms of adding m)

This is analogous to:

- Defining what happens when you add 0 (base)
- Defining what happens when you add “one more,” given you know how to add the previous amount (recursive)

7.3 Definition of Addition

For any $n, m \in \mathbb{N}$, we define $n + m$ by recursion on m :

7.3.1 Base Case

$$n + 0 = n$$

Meaning: Adding zero to anything gives that thing back.

Why this choice?

- Makes 0 the “additive identity”
- Matches intuition: “3 apples plus 0 apples = 3 apples”
- Provides the base for our recursion

Important: This is a **definition**, not a theorem from any axiom. We **choose** to define addition this way.

7.3.2 Recursive Case

$$n + S(m) = S(n + m)$$

Meaning: To add n to the successor of m , take the successor of $(n + m)$.

In words: “Adding to one-more-than- m is the same as doing one-more-than (adding to m)”

Why this choice?

- Breaks down $n + S(m)$ into a simpler problem: $n + m$
- We “peel off” the S from the right, wrap it around the result
- Matches intuition: $3 + 5 = 3 + (4 + 1) = (3 + 4) + 1$

7.4 How the Definition Works

7.4.1 Example 1: Computing $3 + 2$

Notation:

- $1 = S(0)$
- $2 = S(S(0)) = S(1)$
- $3 = S(S(S(0))) = S(2)$

Computation:

$$\begin{aligned} 3 + 2 &= 3 + S(1) && [\text{because } 2 = S(1)] \\ &= S(3 + 1) && [\text{recursive rule: } n + S(m) = S(n + m)] \\ &= S(3 + S(0)) && [\text{because } 1 = S(0)] \\ &= S(S(3 + 0)) && [\text{recursive rule again}] \\ &= S(S(3)) && [\text{base case: } n + 0 = n] \\ &= S(S(S(S(S(0)))))) && [\text{because } 3 = S(S(S(0)))] \\ &= 5 && [\text{by definition of 5}] \end{aligned}$$

What happened:

1. We applied the recursive rule repeatedly
2. Each time, we “peeled off” one S from the right
3. We “wrapped” that S around the outside
4. Eventually we reached $+0$, where we used the base case
5. We ended up with the right number of S ’s wrapped around 0

7.4.2 Example 2: Computing $0 + 3$

Is this the same as $3 + 0$?

$$\begin{aligned} 0 + 3 &= 0 + S(2) && [\text{because } 3 = S(2)] \\ &= S(0 + 2) && [\text{recursive rule}] \\ &= S(0 + S(1)) && [\text{because } 2 = S(1)] \\ &= S(S(0 + 1)) && [\text{recursive rule}] \\ &= S(S(0 + S(0))) && [\text{because } 1 = S(0)] \\ &= S(S(S(0 + 0))) && [\text{recursive rule}] \\ &= S(S(S(0))) && [\text{base case}] \\ &= 3 && [\text{by definition}] \end{aligned}$$

So $0 + 3 = 3$

Notice: We needed to use the recursive rule 3 times, then the base case.

Compare with $3 + 0$:

$$3 + 0 = 3 \quad [\text{IMMEDIATELY by base case!}]$$

This is much simpler!

7.5 A Shocking Truth

From the definition alone:

- We **know** $n + 0 = n$ (that's the base case—part of the definition)
- We **don't immediately know** $0 + n = n$ (must prove as a theorem!)
- We **don't immediately know** $S(n) + m = S(n + m)$ (must prove!)
- We **don't immediately know** $m + n = n + m$ (must prove!)

Even things that feel “obvious” must be proven from the definition!

This is the essence of rigorous mathematics: we prove even the “obvious” things.

7.6 What's Next

In the continuation of this treatment, we will:

1. **Prove** $0 + n = n$ (left identity) — surprisingly non-trivial!
2. **Prove** $S(m) + n = S(m + n)$ (left successor property) — harder!
3. **Prove** $m + n = n + m$ (commutativity) — very hard, requires previous results!
4. **Prove** $(a + b) + c = a + (b + c)$ (associativity)
5. **Define multiplication** recursively
6. **Prove properties** of multiplication (commutativity, associativity, distributivity)
7. **Define order relation** \leq
8. **Prove well-ordering principle**

Each theorem builds on previous results, forming a logical hierarchy from the simple Peano Axioms to the full arithmetic structure.

Coming soon!